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Combinatorial Sums $\sum_j \binom{n}{mj+q}$ Associated with Chebyshev Polynomials

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1.

In this paper we generalize results of Hoggatt and Alexanderson [1] concerning the computation of combinatorial sums of the form

$$S_n(m, q) = \sum_j \binom{n}{mj+q}, \qquad q = 0, 1, \ldots, m-1.$$

These sums arise in a variety of circumstances:

EXAMPLE 1 (Fibonacci roulette, [2]). Let N be a fixed positive integer, preferably large. Figure 1 shows the playing board of a game which operates as follows. As a fair coin is tossed, the ball moves one position clockwise when heads appear, and does not move when tails appear. The ball begins at 0, and after N tosses the payoff is determined by the final position of the ball.

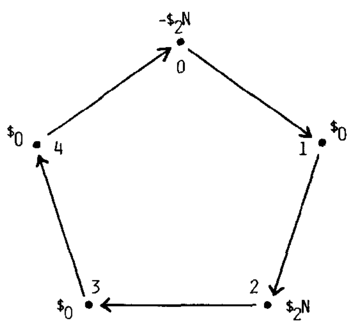

FIGURE 1

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The expected value of the game is easily verified to be

$$\$\left(\sum_j \binom{N}{5j+2} - \sum_j \binom{N}{5j}\right) \tag{1}$$

which may be positive, negative, or zero, depending on N. In fact, if not zero and apart from sign, it is (see below) always a Fibonacci number.

EXAMPLE 2 (a polygon iteration, [3]). In this well-known problem, a generalized m-gon is constructed in the plane by drawing straight lines in unbroken order, with the last edge connecting back to the starting point. Figure 2a shows one. An iterative process is begun by connecting the mid-points of succeeding edges, thus forming a new configuration. The process is then repeated on this polygon, ad infinitum. During the course of the iterations, the polygons can (and must) be rescaled to prevent them from shrinking to a point. The outcome: with a randomly drawn initial polygon, the resuting configurations eventually unwrap, become convex, become elliptic, and stabilize (Fig. 2b).

In vector notation the kth point of the Nth iterated polygon ($0 < s_N =$ scaling factor) is

$$V_k^{(N)} = 2^{-N} s_N \sum_{q=0}^{m-1} V_{k+q}^{(0)} \sum_j \binom{N}{mj+q}.$$

EXAMPLE 3 (a classical identity [1, 4]). Hoggatt and Alexanderson [1] used identity (2), below, to compute $S_n(m, q)$ for selected values of m. They found, for example, the representation of $S_n(5, q)$ in terms of

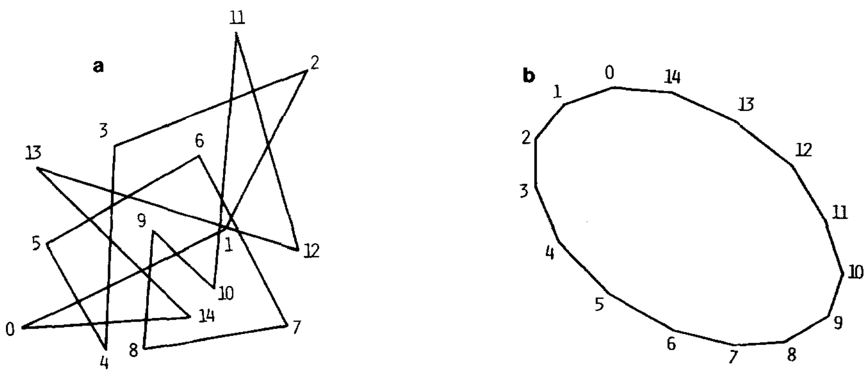

FIGURE 2

Fibonacci numbers. We will provide the framework for extending their results to all values of m.

$$\sum_j \binom{N}{mj+q} = \frac{1}{m} \sum_{j=0}^{m-1} \left(2 \cos \frac{j\pi}{m}\right)^N \cos\left(\frac{(N-2q)\, j\pi}{m}\right). \tag{2}$$

2

Identity (2), due to Ramus [4], has an ambivalent quality. While it is quite satisfactory for attacking a problem like Example 2, it nonetheless expresses a combinatorial, integer value on the left in terms of transcendental, analytic objects on the right. In a problem like Example 1, the expression of (1) in terms of classical counting numbers seems more natural. Yet, when it comes to the sums $S_n(m, q)$, even extensive treatises like [5] and [6] feature essentially no more than (2). In [5] it is Problem 7 of Chapter 2, while [6, Sect. 4.3], discusses some associated methodology.

We will show that the Chebyshev polynomials (second kind) supply the missing information. In fact, these polynomials and the family of games suggested by Example 1 are intimately related, a claim substantiated in Section 4, below.

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We use a method of generalized binomial coefficients (*gbc*'s) [2] to find the analogs in $S_n(m, q)$ of the Fibonacci numbers, $m = 1, 2,...$, as well as giving a combinatorial development of the Chebyshev polynomials.

In [2] we defined three types of *gbc*'s, so named because of their close adherence to the Pascal recurrence. They were

$$\begin{aligned} &\left\{ {n+1 \atop j} \right\} = \left\{ {n \atop j} \right\} + \left\{ {n \atop j+1} \right\}, \qquad n = 0, 1, 2,..., \quad ..., -2, -1, 0, 1, 2,... = j, \\ &\left[{n \atop j} \right] = (-1)^n \left\{ {n \atop j} \right\}, \\ &\left\langle {n \atop j} \right\rangle = \left[{n \atop j} \right] - \inf_i \left[{n \atop i} \right]. \end{aligned} \tag{3}$$

The initial values $\{ {0 \atop j} \}$ are, aside from the requirement $\sup_j |\{ {0 \atop j} \}| < +\infty$, chosen freely. Equation (4), with $r = i = 0$, shows that the coefficients $\{ {n \atop 0} \}$ evaluate entire binomial sums. Essentially all the information of [2] is summarized in

THEOREM 1. *The binomial coefficients* $\binom{n}{j}$ *and gbc's defined in* (3) *satisfy*

$$\sum_j \binom{n}{j} \left\{ {r \atop j+i} \right\} = \left\{ {n+r \atop i} \right\}, \qquad \textit{for } n, r \geqq 0 \textit{ and any integer } i, \tag{4}$$

$$\left\langle {n+1 \atop j} \right\rangle = \lambda_{n+1} - \left\langle {n \atop j} \right\rangle - \left\langle {n \atop j+1} \right\rangle,$$

$$\textit{with } \lambda_{n+1} = \sup_i \left(\left\langle {n \atop i} \right\rangle + \left\langle {n \atop i+1} \right\rangle \right), \tag{5}$$

$$\left\{ {n \atop j} \right\} = (-1)^n \left\langle {n \atop j} \right\rangle + \sum_{i=1}^{n} (-1)^{i-1} 2^{n-i} \lambda_i,$$

$$\left[{n \atop j} \right] = \left\langle {n \atop j} \right\rangle + \sum_{i=1}^{n} (-1)^{n+i-1} 2^{n-i} \lambda_i,$$

$$\sum_j \binom{n}{j} \left\langle {r \atop j+i} \right\rangle = (-1)^n \left\langle {n+r \atop i} \right\rangle + \sum_{j=1}^{n} (-1)^{j-1} 2^{n-j} \lambda_{r+j}. \tag{6}$$

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The feature of (6) which makes it fundamentally different from (2) is that it is in the tradition of counting the same thing in two different ways. On the left in (6), with $r = i = 0$, we sum across the nth row of Pascal's triangle with each binomial coefficient multiplied by an arbitrary number $\langle {0 \atop j} \rangle$. On the right, we obtain the same value by moving up the rows, including and excluding the appropriately weighted (by λ_j) row sums $\sum_{i=0}^{n-j} \binom{n-j}{i} = 2^{n-j}$, $j = 1, 2,\ldots$, plus a residual $(-1)^n \langle {n \atop 0} \rangle$. Recurrence (5) is used to generate the Pascal-like array of *gbc*'s $\langle {n \atop j} \rangle$, once an initial row is given.

For purposes of illustration, let this initial row be defined by $\langle {0 \atop j} \rangle = 1$ if $j \equiv 0 \pmod 5$ and zero otherwise. The resulting array is

$$\begin{array}{ccc|ccccc|cc} & & & j=0 & & & & & & \\ n=0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \\ & & 2 & 1 & 0 & 0 & 1 & 2 & 1 & \\ & & 0 & 2 & 3 & 2 & 0 & 0 & 2 & \\ & & 3 & 0 & 0 & 3 & 5 & 3 & 0 & \\ & \cdots & 5 & 8 & 5 & 0 & 0 & 5 & 8 & \cdots \\ & & & \vdots & & & & \vdots & & \end{array} \tag{7}$$

Array (7) shows that each row consists of repeating *blocks* of length 5. Observe that each block has the form $(b_n, a_n, 0, 0, a_n)$ modulo a shift with wraparound, and after the first few rows $0 < a_n < b_n$. The generating algorithm (5) implies

$$b_{n+1} = b_n + a_n$$

$$a_{n+1} = b_{n+1} - a_n,$$

consequently $b_{n+2} = b_{n+1} + b_n$ and $a_{n+2} = a_{n+1} + a_n$. In other words, both sequences $\{b_n\}_{n \geqq 0}$ and $\{a_n\}_{n \geqq 0}$ satisfy the Fibonacci recurrence and are, in fact, the Fibonacci numbers. Therefore we have the association

$$\sum_j \binom{n}{5j+q} \leftrightarrow Q_2(x) = x^2 - x - 1,$$

where Q_2 is the characteristic polynomial of the Fibonacci recurrence. Array (7) can be used in conjunction with (6) to give an elegant proof that (1) is either zero or (+ or −) a Fibonacci number, as well as a compact representation of the information obtained by Hoggatt and Alexanderson on $S_n(5, q)$.

The point to be emphasized is that the foregoing calculation could have been carried out for *any* modulus $m = 2p + 1$ to find the association

$$\sum_j \binom{n}{mj+q} \leftrightarrow Q_p(x).$$

Now for the connection with $\{U_n\}_{n \geqq 0}$, the Chebyshev polynomials of the second kind,

$$Q_p(x) = U_p(x/2) - U_{p-1}(x/2), \qquad p = 0, 1, \ldots$$

This supports our claim about the correspondence between the family of games suggested by Example 1 and $\{U_n\}_{n \geqq 0}$. Let $m = 2p + 1$, and let the five positions of Fig. (1) be replaced with m positions equipped with payoffs $2^N \pi_j$. Suppose $\pi_0 + \cdots + \pi_{m-1} = 0$. Then (6) gives the expected value

$$\sum_{q=0}^{m-1} \pi_q \sum_j \binom{N}{mj+q} = (-1)^N \sum_{q=0}^{m-1} \left\langle \begin{matrix} N \\ -q \end{matrix} \right\rangle^* \pi_q,$$

where $(\langle {0 \atop q} \rangle^*)_{q=0}^{m-1} = (1, 0, 0, \ldots, 0)$. Theorem 2, below, states precisely how the values $\langle {N \atop \cdot} \rangle^*$ obey the recurrence whose characteristic polynomial is Q_p. Of course, similar results hold when $m = 2p$ is an even integer. The complete story is given in

THEOREM 3. *Let the initial row of an array of gbc's* $\langle {n \atop j} \rangle$ *consist of blocks* $B_0 = (1, 0, 0, \ldots, 0)$ $(m-1$ *zeroes). Let* $m = 2p+1$, $p = 0, 1, \ldots,$ *and* $R_{(\cdot)}$ *denote the right shift operator with wraparound. Then, the* nth *row block* B_n *of the array of gbc's has the form*

$$M_n = (\ldots, x_n, \ldots, c_n, b_n, a_n, 0, 0, a_n, b_n, c_n, \ldots),$$

where eventually $0 < a_n < b_n < c_n \cdots < x_n < \cdots$ *and* $B_n = R_{pn(\operatorname{mod} m)} M_n$. *The sequences* $\{x_n\}_{n \geqq 0}$ *satisfy the linear recurrence whose characteristic polynomial is*

$$Q_p(x) = U_p(x/2) - U_{p-1}(x/2).$$

Let $m = 2p$, $p = 1, 2, \ldots,$ *and* $L_{(\cdot)}$ *denote the left shift operator with wraparound. Then, the even row blocks* $\bar{B}_{2n}$ *have the form*

$$\bar{M}_{2n} = (\ldots, \bar{x}_{2n}, \ldots, \bar{c}_{2n}, \bar{b}_{2n}, \bar{a}_{2n}, 0, \bar{a}_{2n}, \bar{b}_{2n}, \bar{c}_{2n}, \ldots),$$

while the odd row blocks $\bar{B}_{2n+1}$ *have the form*

$$\bar{M}_{2n+1} = (0, \bar{a}_{2n+1}, \bar{b}_{2n+1}, \ldots, \bar{x}_{2n+1}, \ldots \ \ldots, \bar{x}_{2n+1}, \ldots, \bar{b}_{2n+1}, \bar{a}_{2n+1}, 0),$$

where eventually $0 < \bar{a}_k < \bar{b}_k < \bar{c}_k < \cdots < \bar{x}_k < \cdots$ *and* $\bar{B}_k = L_{[k/2](\operatorname{mod} m)} \bar{M}_k$. *The even and odd subsequences* $\{\bar{x}_{2n}\}_{n \geqq 0}$ *and* $\{\bar{x}_{2n+1}\}_{n \geqq 0}$ *satisfy the linear recurrence whose characteristic polynomial is*

$$\bar{Q}_p(x) = U_{p-1}((x-2)/2).$$

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